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Ambiguities Appearing in the Study of Time-Dependent Constants of Motion for the One-Dimensional Harmonic Oscillator

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A family of time-dependent constants of motion for the one-dimensional harmonic oscillator is derived. The relation between constants of motion, Lagrangian, and Hamiltonian is described. A well-defined time-dependent Lagrangian (for which Euler–Lagrange equations and Legendre transformation are fully satisfied) is not uniquely determined.

1. INTRODUCTION

The concept of "constant of motion" has brought about important relations between the Lagrangian and the Hamiltonian formalism in classical dynamical systems (López, 1993a, b, 1996a; López and Hernández, 1987; Dodonov *et al.*, 1981; Goldstein, 1980) and has had application even in heat conduction theory (López, 1996b). A constant of motion for an autonomous system [where the forces do not depend explicitly on time (Drazin, 1992)] does not need to be time dependent. However, for nonautonomous systems (where the forces depend explicitly on time), the constant of motion must be time dependent. For Hamiltonian systems (Goldstein, 1980) one normally does not worry whether or not the Hamiltonian should be a constant of motion. The application of time-dependent Hamiltonians to the study of classical and quantum harmonic oscillators has been rather extensive (Dekker, 1981, and references therein). Of particular interest in quantum mechanics are one-dimensional systems (Zel'dovich, 1967; Man'ko and Haake, 1992; Croxson, 1994; Camiz *et al.*, 1971; Kim and Man'ko 1991), where an explicitly time-dependent Hamiltonian is used for different studies. This same

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Hamiltonian formalism has many attractions for the study of chaotic behavior in classical (Chirikov, 1979) and quantum (Leyvraz and Seligman, 1992) systems. In fact, one needs at least a one-dimensional nonautonomous system (so-called 1.5 dynamical system) to see chaotic behavior (Ott, 1994). Therefore, the constants of motion of one-dimensional nonautonomous systems may be relevant to chaotic systems and to the relation among the concepts "constant of motion," "Lagrangian," and "Hamiltonian."

The relation among time-dependent constants of motion, Lagrangian, and Hamiltonian for nonautonomous systems has already pointed out elsewhere (López and Hernández, 1987). In this analysis the problem of a time-dependent constant of motion for an autonomous system was also considered. The present paper is focused on this last problem specifically for the classical harmonic oscillator in one dimension.

2. TIME-DEPENDENT CONSTANT OF MOTION

The one-dimensional harmonic oscillator can be written as the dynamical system

$$\frac{dx}{dt} = v \tag{1a}$$

and

$$\frac{dv}{dt} = -\omega^2 x \tag{1b}$$

where ω is the angular frequency, x is the position, and v is the velocity. A time-dependent constant of motion for this system is a function $K(x, v, t)$ which satisfies the following partial differential equation:

$$v\frac{\partial K}{\partial x} - \omega^2 x\frac{\partial K}{\partial v} + \frac{\partial K}{\partial t} = 0 \tag{2}$$

This equation can be solved by the characteristics method (Sneddon, 1957), where the equations for the characteristics (C_1 and C_2) are

$$\frac{dx}{v} = \frac{dv}{-\omega^2 x} = \frac{dt}{1} = \frac{dK}{0} \tag{3}$$

The general solution of equation (2) will be an arbitrary function of these characteristics, $K(x, v, t) = G(C_1, C_2)$. The first characteristics can be obtained by integrating the first two terms of (3), which brings about the characteristic

$$C_1 = \frac{1}{2}mv^2 + \frac{1}{2}\omega^2x^2 \tag{4}$$

From this expression we obtain $v = v(x, C_1)$, which can be used in equation (3) together with the third term to obtain the second characteristic

$$C_2 = \arcsin\sqrt{\frac{\omega^2x^2}{v^2 + \omega^2x^2}} - wt \tag{5}$$

Note that the units of equations (4) and (5) are respectively "energy" and "none." Therefore, one can select as the general solution of (2) the following

$$K(x, v, t) = G(g_1(C_1) \cdot g_2(C_2)) \tag{6}$$

where the functions g_1, g_2 and G are arbitrary. In particular, one may chose the solution

$$K(x, v, t) = \sin(C_2)C_1^n \tag{7}$$

where n is an arbitrary positive integer, i.e., the function K has the form

$$K(x, v, t) = \left(\frac{m}{2}\right)^2(v^2 + \omega^2x^2)^{n-1/2}\,[\omega x\cos(\omega t) - v\sin(\omega t)] \tag{8}$$

This family of explicitly time-dependent constants of motion already represents an ambiguity for determining the proper constant of motion of the system (1).

3. RELATION BETWEEN CONSTANT OF MOTION AND LAGRANGIAN

The relationship between Hamiltonians and Lagrangians is given by the Legendre transformation $vp - L = H$, where p is called the generalized linear momentum and is given by $p = \partial L/\partial v$. The Lagrangian is a function of x, v and t, $L = L(x, v, t)$, and the Hamiltonian is a function of x, p and t, $H = H(x, p, t)$. If this transformation is seen as a function of (x, p, t), then one needs to know $v = v(x, p, t)$ and substitute it in this transformation. On the other hand, seeing the Legendre transformation as a function of (x, v, t), one needs to substitute $p = p(x, v, t)$. Doing this in the Hamiltonian, one defines a new function $K(x, v, t) = H(x, p(x, v, t), t)$, and the Legendre transformation can be seen as the following partial differential equation:

$$v \frac{\partial L}{\partial v} - L = K(x, v, t) \tag{9}$$

which has the solution (López and Hernández, 1987)

$$L(x, v, t) = A(x, t)\, v + v \int^{v} \frac{K(x, \xi, t)}{\xi^2}\, d\xi \tag{10}$$

The first term on the right side of equation (10) represents the "gauge" of the Lagrangian and is the solution of the homogeneous equation associated to (10) ($K = 0$), which can be ignored. (However, this term may have a contribution to the Euler–Lagrange equations if $\partial A/\partial t \neq 0$). Now, taking the total time derivative of (9), it follows that

$$v \left[\frac{d}{dt} \frac{\partial L}{\partial v} - \frac{\partial L}{\partial x} \right] - \frac{\partial L}{\partial t} = \frac{dK}{dt} \tag{11}$$

This equation and equation (10) form the foundation for understanding the relationship between the constant of motion and Lagrangian.

A. If the function L represents the Lagrangian of the system, i.e., L satisfies the Euler–Lagrange equation,

$$\frac{d}{dt} \frac{\partial L}{\partial v} - \frac{\partial L}{\partial x} = 0 \tag{12a}$$

which generates the equations of motion, then from (11) it follows that

$$-\frac{\partial L}{\partial t} = \frac{dK}{dt} \tag{12b}$$

Therefore, if $\partial L/\partial t$ is different from zero, the function K cannot be a constant of motion. If $\partial L/\partial t$ is equal to zero (so that L does not depend explicitly on time), then the function K must be a constant of motion. But from equation (10), this constant of motion cannot depend explicitly on time, otherwise the Lagrangian would depend explicitly on time. So, one can say that explicitly time-dependent Lagrangians yield functions K (Hamiltonians) which are not constants of motion.

B. If the function L does not depend explicitly on time ($\partial L/\partial t = 0$), equation (11) yields the equation

$$v \left[\frac{d}{dt} \frac{\partial L}{\partial v} - \frac{\partial L}{\partial x} \right] = \frac{dK}{dt} \tag{12c}$$

which indicates (for $v \neq 0$) that the function K is a constant of motion if and only if the function L is the Lagrangian of the system, so equation (12a) is satisfied. Furthermore, due to equation (10), this constant of motion does not depend on time. So, one can say that the explicitly time-independent Lagrangian of the system brings about a function K (Hamiltonian) which is a constant of motion.

C. If the function L depends explicitly on time ($\partial L/\partial t \neq 0$), then due to equation (10), the function K must depend explicitly on time ($\partial K/\partial t \neq 0$). If this function K is a constant of motion ($dK/dt = 0$), then the function L cannot be the Lagrangian of the system since equation (12a) is not satisfied.

The conclusions obtained from A–C point out that if one finds an explicitly time-dependent constant of motion for the dynamical system, and this is the function appearing on the right side of equation (9), then the Lagrangian of the system must not exist, since a contradiction between equations (10) and (11) would appear. Therefore, an explicitly time-dependent constant of motion should not be used in equation (9) if the Lagrangian of the system exists.

4. FUNCTION L FOR THE EXPLICITLY TIME-DEPENDENT CONSTANT OF MOTION

Using the time-dependent constant of motion (8) into equation (10) gives

$$L = \left(\frac{m}{2}\right)^n \omega x v \cos(\omega t) \int^v \frac{(\xi^2 + \omega^2 x^2)^{n-1/2}}{\xi^2}\, d\xi - \left(\frac{m}{2}\right)^n v \sin(\omega t) \int^v \frac{(\xi^2 + \omega^2 x^2)^{n-1/2}}{\xi}\, d\xi$$

which can readily be integrated (Gradshteyn and Ryzhik, 1994), giving the following expression:

$$\begin{aligned} L = {} & \left(\frac{m}{2}\right)^n \omega x v \cos(\omega t) \\ & \times \left\{ -\frac{(v^2 + \omega^2 x^2)^{n-1/2}}{v} + \frac{(2n-5)!!\,(2\omega x)^{2n-1}}{8^{n-1}\,(n-1)!}\, Arsh\left(\frac{v}{\omega x}\right) \right. \\ & + \frac{v\sqrt{v^2 + \omega^2 x^2}\,(2n-1)}{2(n-1)} \left[(v^2 + \omega^2 x^2)^{n-2} \right. \\ & \left.\left. + \sum_{k=0}^{n-3} \frac{(2n-3)(2n-5)\ldots(2n-2k-3)\,(2\omega x)^{2k+1}\,(v^2 + \omega^2 x^2)^{n-k-3}}{8^{k+1}\,(n-2)\,(n-3)\ldots(n-k-2)} \right] \right\} \end{aligned}$$

$$- \left(\frac{m}{2}\right)^n v \sin(\omega t) \left\{ \sum_{k=0}^{n-2} \frac{(\omega x)^{2k}(v^2 + \omega^2 x^2)^{n-k-1/2}}{2(n-k)-1} + (\omega x)^{2n-2}\sqrt{v^2 + \omega^2 x^2} - (\omega x)^{2n-1} \sinh^{-1}\left(\frac{\omega x}{v}\right) \right\} \tag{14a}$$

where $n \geq 2$, and the function $\sinh^{-1}(z)$ is defined as

$$\sinh^{-1}(z) = \log\left(z + \sqrt{z^2 + 1}\right) \tag{14b}$$

For $n = 1$, the function L is given by

$$L = \frac{m\omega x v}{2} \cos(\omega t) \left[-\frac{\sqrt{v^2 + \omega^2 x^2}}{v} + \sinh^{-1}\left(\frac{v}{\omega x}\right) \right] - \frac{mv}{2} \sin(\omega t) \left[\sqrt{v^2 + \omega^2 x^2} - \omega x \sinh^{-1}\left(\frac{\omega x}{v}\right) \right] \tag{15}$$

The functions (14a) and (15) do not represent the Lagrangian of the system (1) since they do not satisfy the Euler–Lagrange equation (12a), but the equation

$$v \left[\frac{d}{dt}\frac{\partial L}{\partial v} - \frac{\partial L}{\partial x} \right] - \frac{\partial L}{\partial t} = 0$$

as can be readily demonstrated.

CONCLUSION

A family of time-dependent constants of motion is found, and the relation between this type of constant of motion and the Lagrangians and Hamiltonians was studied through the Legendre transformation and the Euler–Lagrange equation. The results indicate that a well-defined time-dependent Lagrangian (one for which the Euler–Lagrange equation and the Legendre transformation are fully satisfied) cannot be found. However, one can find the explicitly time-dependent function L, equations (14a) and (15), for the harmonic oscillator.

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